

Penrose Limit of $\mathcal{N} = 1$ Gauge Theories

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We find a Penrose limit of $AdS_5 \times T^{1,1}$ which gives the pp-wave geometry identical to the one that appears in the Penrose limit of $AdS_5 \times S^5$. This leads us to conjecture that there is a subsector of the corresponding $\mathcal{N} = 1$ gauge theory which has enhanced $\mathcal{N} = 4$ supersymmetry. We identify operators in the $\mathcal{N} = 1$ gauge theory with stringy excitations in the pp-wave geometry and discuss how the gauge theory operators fall into $\mathcal{N} = 4$ supersymmetry multiplets. We find similar enhancement of symmetry in some other models, but there are also examples in which there is no supersymmetry enhancement in the Penrose limit.

1. Introduction

The AdS/CFT correspondence [1,2,3] relates a conformal field theory in $(p+1)$ -dimensions to string theory in $AdS_{p+2} \times X$, where X is a compact Einstein space. In the last few years, we have learned much about nonperturbative aspects of string theory and conformal field theories using this correspondence [4]. One of the major obstructions in making further progress in this direction has been our lack of understanding of the worldsheet dynamics describing string theory in AdS and related backgrounds. Understanding this problem is essential, for example, to finding quantitative results from string theory for the large N limit of gauge theories with finite 't Hooft coupling. Although much progress has been made for string theory in AdS_3 with an NS-NS background [5], worldsheet dynamics in higher dimensional AdS spaces and/or with R-R field strengths remains a mystery. Thus far, most of the results obtained from string theory in AdS have relied on the supergravity approximation.

Recently, it was shown in [6] that the worldsheet theory of the Type IIB string on the maximally supersymmetric pp-wave geometry [7]:

$$ds^2 = -4dx^+dx^- + \sum_{i=1}^8 (dr^i dr^i - r^i r^i dx^+ dx^+), \quad (1.1)$$

with constant R-R 5-form flux,

$$F_{+1234} = F_{+5678} = \text{const}, \quad (1.2)$$

is exactly soluble in the light-cone Green-Schwarz formalism [8,9]. Moreover, in a recent interesting paper [10], it was pointed out that Type IIB string theory on this pp-wave background is dual to the large N limit of a certain subsector of four dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory. The subsector is characterized by choosing a $U(1)_R$ subgroup of the $SU(4)_R$ R-symmetry of the gauge theory and by considering states with conformal weight Δ and $U(1)_R$ charge R which scale as $\Delta, R \sim \sqrt{N}$ and whose difference $(\Delta - R)$ is finite in the large N limit. The claim is that, in the $N \rightarrow \infty$ limit with the string coupling $g_s = g_{YM}^2$ finite, the subspace of the gauge theory Hilbert space and the operator algebra preserving these conditions are described by string theory in the pp-wave geometry. This duality was derived in [10] by starting with the familiar correspondence between $\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory and Type IIB string theory in $AdS_5 \times S^5$ and considering a scaling limit of the geometry near a null geodesic in $AdS_5 \times S^5$ carrying

large angular momentum with respect to the $U(1)_R$ isometry of S^5 . This corresponds to truncating to the appropriate subsector of the gauge theory in the scaling limit. The string theory background that one obtains in the scaling limit is the pp-wave geometry with a constant R-R flux which can then be quantized in the light-cone gauge [6]. The scaling limit is a special example of the Penrose limit which transforms any solution of supergravity to a plane wave geometry [11,12,13,14].

In [10], it was shown that operators in the appropriate subsector of $\mathcal{N} = 4$ $SU(N)$ gauge theory can be identified with *stringy* oscillators in the pp-wave background. This matching makes quantitative predictions about the spectrum of the gauge theory beyond the supergravity approximation, and some of them were checked in [10] using gauge theory computation of planar Feynman diagrams.

In this paper we consider a similar duality that exists between a certain four-dimensional $\mathcal{N} = 1$ gauge theory and Type IIB string theory in a pp-wave background. We derive this duality by taking a scaling limit of the duality [15] between Type IIB string theory on $AdS_5 \times T^{1,1}$ and the four-dimensional superconformal field theory which consists of an $\mathcal{N} = 1$ $SU(N) \times SU(N)$ super Yang-Mills multiplet with a pair of bifundamental chiral multiplets A_i and B_i transforming in the (N, \bar{N}) and (\bar{N}, N) representation of the gauge group. The gauge theory is flown to the IR fixed point and deformed by an $SU(2)_1 \times SU(2)_2$ invariant superpotential

$$W = \frac{\lambda}{2} \epsilon^{ij} \epsilon^{i'j'} \text{Tr} A_i B_{i'} A_j B_{j'}, \quad (1.3)$$

which is exactly marginal at the fixed point. This gives the theory of [15] that lives¹ on N D3-branes sitting at the conifold singularity of a Calabi-Yau three-fold. The scaling limit is obtained by considering the geometry near a null geodesic carrying large angular momentum in the $U(1)_R$ isometry of the $T^{1,1}$ space which is dual to the $U(1)_R$ R-symmetry in the $\mathcal{N} = 1$ superconformal field theory.

The scaling limit around this null geodesic in $AdS_5 \times T^{1,1}$ results in a pp-wave geometry. We identify the light-cone Hamiltonian, longitudinal momentum and angular momentum of string theory in the pp-wave geometry with linear combinations of the conformal weight Δ , $U(1)_R$ charge R and $U(1)_1 \times U(1)_2$ global charges Q_1 and Q_2 of operators in the dual $\mathcal{N} = 1$ gauge theory. The appropriate scaling limit requires truncating the gauge

¹ The details of the gauge theory will appear in section 3.

theory Hilbert space to those operators whose conformal weight Δ , R-charge R and global charges Q_1 and Q_2 scaling like

$$\Delta, R, Q_1, Q_2 \sim \sqrt{N}, \quad (1.4)$$

with

$$\Delta - \frac{3}{2}R, Q_1 - \frac{1}{2}R, \text{ and } Q_2 - \frac{1}{2}R : \text{ finite}, \quad (1.5)$$

in the large N limit.

Remarkably, the pp-wave geometry that one obtains in the scaling limit can be transformed into the maximally supersymmetric background of (1.1) and (1.2) after a suitable change of coordinates. Therefore, in this limit supersymmetry is enhanced. This implies that the subsector of the Hilbert space of the $\mathcal{N} = 1$ gauge theory of [15] obeying the conditions (1.4) and (1.5) has a hidden $\mathcal{N} = 4$ supersymmetry. We believe that this is a very interesting prediction of our duality that deserves further study.

We find that the change of coordinates induces *twisting* of the light-cone Hamiltonian of the string theory by

$$p^- = p_{S^5}^- + J_1 + J_2, \quad (1.6)$$

where $p_{S^5}^-$ is the Hamiltonian of the maximally supersymmetric wave found in [6] and J_1, J_2 correspond to rotation charges under an $\mathbf{R}^2 \times \mathbf{R}^2$ subspace of the transverse space of the pp-wave geometry. From the gauge theory point of view, the light-cone Hamiltonian p^- before the twisting is $\Delta - \frac{3}{2}R$ and the rotational charges are given by $J_a = Q_a - \frac{1}{2}R$ ($a = 1, 2$). Note that they remain finite in the limit (1.5). After the twisting, the light-cone Hamiltonian is identified with $\Delta_{\mathcal{N}=4} - R_{\mathcal{N}=4}$, in terms of the conformal weight and the R charge for the $\mathcal{N} = 4$ supersymmetry algebra. Thus we find the following relation

$$\begin{aligned} \Delta_{\mathcal{N}=4} - R_{\mathcal{N}=4} &= \Delta - \frac{3}{2}R - J_1 - J_2 \\ &= \Delta - \frac{1}{2}R - Q_1 - Q_2. \end{aligned} \quad (1.7)$$

The spectrum of stringy excitations in the $\mathcal{N} = 4$ theory studied in [10] can then be turned into that of the $\mathcal{N} = 1$ theory by this twisting. The twisted string spectrum is highly degenerate, and we show that it matches with gauge theory expectations.

The Penrose limit focuses on geometry near a null geodesic. When we have a gauge theory whose supersymmetry is reduced by placing branes on a curved space, the Penrose limit may flatten out the space and restore supersymmetry. We have found other examples

where similar enhancement of symmetry takes place. Those include the $\mathcal{N} = 1$ pure super Yang-Mills theory (with Kaluza-Klein tower of fields) studied in [16] and gauge theories realized on branes on a C^3/Z_3 orbifold singularity. The limit of the former is a variation of the Nappi-Witten geometry [17] with 16 supercharges, and that of the latter is the maximally supersymmetric pp-wave. On the other hand, there are cases in which such enhancement does not happen, such as gauge theories on branes at a C^2/Z_2 orbifold singularity.

The rest of the paper is organized as follows. In section 2, we consider the scaling limit around a null geodesic in $AdS_5 \times T^{1,1}$ and show that one obtains a pp-wave background. We identify the subsector of the Hilbert space of the dual superconformal field theory that is dual to string theory in the pp-wave geometry. We show that there is a coordinate transformation which brings the pp-wave background to the one which has maximal supersymmetry (1.1). The Hamiltonian of the pp-wave is then obtained by twisting the Hamiltonian of the Type IIB string in that background by angular momentum charges J_1 and J_2 that the strings carry in the maximally supersymmetric pp-wave background. In section 3, we describe ingredients of the $\mathcal{N} = 1$ theory of [15] that are required to compare the gauge theory and the string theory. Precise matching is obtained by identifying in a specific way string theory excitations with gauge theory operators. In section 4, we discuss other examples in which similar enhancement of symmetry takes place and show an example where symmetry enhancement does not occur. We conclude with a discussion. In the appendix we explicitly solve for the worldsheet theory of the pp-wave background that we obtain in the limit. Explicit diagonalization of the Hamiltonian shows that it is related to that of the maximally supersymmetric wave by twisting.

Note added:

After posting this paper on the e-Print arXiv, we have received [18,19], where the Penrose limit of $AdS_5 \times T^{1,1}$ is studied and the supersymmetry enhancement is also noted. We also received [20], where the Penrose limit of backgrounds with NS-NS 3 form field and its relation to a generalization of the Nappi-Witten model are also discussed.

2. Penrose limit of $AdS_5 \times T^{1,1}$

We start by considering the supergravity solution dual to the $\mathcal{N} = 1$ superconformal field theory of [15] that we describe in the next section. The background of interest is

$AdS_5 \times T^{1,1}$, where $T^{1,1} = (SU(2) \times SU(2))/U(1)$, with the $U(1)$ diagonally embedded in the two $SU(2)$'s. The Einstein metric on $AdS_5 \times T^{1,1}$ is given by

$$\begin{aligned} ds_{AdS}^2 &= L^2 (-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3) \\ ds_{T^{1,1}}^2 &= L^2 \left(\frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \right. \\ &\quad \left. + \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \right), \end{aligned} \quad (2.1)$$

where $d\Omega_3$ is the volume form of a unit S^3 and the curvature radius L of AdS_5 is given by $L^4 = 4\pi g_s N \alpha'^2 27/16$. Topologically, $T^{1,1}$ is a $U(1)$ bundle over $\mathbf{S}^2 \times \mathbf{S}^2$. The base is parametrized by coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2) respectively and the Hopf fiber coordinate ψ has period 4π . The $SU(2)_1 \times SU(2)_2 \times U(1)_R$ isometry group of $T^{1,1}$ is identified with the $SU(2)_1 \times SU(2)_2$ global symmetry and $U(1)_R$ symmetry of the dual superconformal field theory of [15]. In addition, the solution has a constant dilaton and a R-R five-form flux

$$F = L^4 (\text{vol}_{AdS} + \text{vol}_{T^{1,1}}), \quad (2.2)$$

where $\text{vol}_{AdS}, \text{vol}_{T^{1,1}}$ are the volume forms of AdS_5 and $T^{1,1}$.

We now perform a scaling limit around a null geodesic in $AdS_5 \times T^{1,1}$ which rotates along the ψ coordinate of $T^{1,1}$, whose shift symmetry corresponds to the $U(1)_R$ symmetry of the dual superconformal field theory². We introduce coordinates which label the geodesic

$$\begin{aligned} x^+ &= \frac{1}{2} \left(t + \frac{1}{3} (\psi + \phi_1 + \phi_2) \right) \\ x^- &= \frac{L^2}{2} \left(t - \frac{1}{3} (\psi + \phi_1 + \phi_2) \right). \end{aligned} \quad (2.3)$$

and consider a scaling limit around $\rho = \theta_1 = \theta_2 = 0$ in the geometry (2.1). We take $L \rightarrow \infty$ while rescaling the coordinates

$$\rho = \frac{r}{L} \quad \theta_1 = \frac{\sqrt{6}}{L} \xi_1 \quad \theta_2 = \frac{\sqrt{6}}{L} \xi_2. \quad (2.4)$$

² Shifts along the angles ϕ_1 and ϕ_2 generate an $U(1) \times U(1)$ subgroup of the $SU(2)_1 \times SU(2)_2$ isometries and correspond in the gauge theory side to the abelian charges Q_1 and Q_2 , which are the Cartan generators of the $SU(2)_1 \times SU(2)_2$ global symmetry group of the gauge theory.

The metric one obtains in the limit is

$$\begin{aligned}
ds^2 &= -4dx^+dx^- + \sum_{i=1}^4 (dr^i dr^i - r^i r^i dx^+ dx^+) \\
&\quad + \sum_{a=1,2} (d\xi_a^2 + \xi_a^2 d\phi_a^2 - 2\xi_a^2 d\phi_a dx^+) \\
&= -4dx^+dx^- + \sum_{i=1}^4 (dr^i dr^i - r^i r^i dx^+ dx^+) \\
&\quad + \sum_{a=1,2} [dz_a d\bar{z}_a + i(\bar{z}_a dz_a - z_a d\bar{z}_a) dx^+].
\end{aligned} \tag{2.5}$$

In the last line, we introduced complex Cartesian coordinates z_a in lieu of (ξ_a, ϕ_a) . The metric has a covariantly constant null Killing vector $\partial/\partial x^-$ so that it is a pp-wave metric. The pp-wave has a natural decomposition of the \mathbf{R}^8 transverse space into $\mathbf{R}^4 \times \mathbf{R}^2 \times \mathbf{R}^2$, where \mathbf{R}^4 is parametrized by r^i and $\mathbf{R}^2 \times \mathbf{R}^2$ by z_a . The geometry is supported by a null, covariantly constant flux of the R-R field,

$$F_{+1234} = F_{+5678} = \text{const.} \tag{2.6}$$

The obvious symmetries of this background are the $SO(4)$ rotations in \mathbf{R}^4 and a $U(1) \times U(1)$ symmetry³ rotating $\mathbf{R}^2 \times \mathbf{R}^2$. In the gauge theory side, the $SO(4)$ symmetry corresponds to the subgroup of the $SO(2,4)$ conformal symmetry (*i.e.* the rotations of S^3 in the field theory space $\mathbf{R} \times S^3$) and the $U(1) \times U(1)$ rotation charges J_1 and J_2 with the $U(1) \times U(1)$ symmetry $Q_1 - \frac{1}{2}R$ and $Q_2 - \frac{1}{2}R$ respectively, where R is the $U(1)_R$ charge of the gauge theory and Q_1, Q_2 are the Cartan generators of the $SU(2)_1 \times SU(2)_2$ global symmetry of the dual superconformal field theory.

In order to compare string theory in the pp-wave geometry (2.5) with the appropriate subsector of the dual field theory determined by the limit, (2.3) and (2.4), we need to establish the correspondence between conserved charges in string theory and in gauge theory. In string theory, the light-cone momenta can be identified with combinations of the conformal weight Δ and the $U(1)_R$ charge R of the dual superconformal field theory by noting that⁴

$$\begin{aligned}
2p^- &= i\partial_{x^+} = i(\partial_t + 3\partial_\psi) = \Delta - \frac{3}{2}R \\
2p^+ &= \frac{i}{L^2}\partial_{x^-} = \frac{i}{L^2}(\partial_t - 3\partial_\psi) = \frac{1}{L^2}\left(\Delta + \frac{3}{2}R\right).
\end{aligned} \tag{2.7}$$

³ In fact, the metric and the flux is invariant under a larger symmetry as we shall see below.

⁴ The factor of two in the normalization of R is due to the 4π periodicity of the ψ coordinate.

The J_1 and J_2 rotation charges of the string can be identified with

$$J_a = -i \frac{\partial}{\partial \phi_a} \Big|_{x^\pm} = -i \frac{\partial}{\partial \phi_a} \Big|_{t, \psi} + i \frac{\partial}{\partial \psi} \Big|_{t, \phi_i} = Q_a - \frac{1}{2} R \quad a = 1, 2 \quad (2.8)$$

such that the states of the dual gauge theory are also labeled by these global symmetry charges.

Therefore, it follows from the identification (2.7) and the $L \rightarrow \infty$ limit, (2.3) and (2.4), that string theory in the pp-wave background (2.5) with finite p^-, p^+ and J_i is dual to the $\mathcal{N} = 1$ gauge theory of [15] in a subsector of the Hilbert space where $\Delta, R, Q_a \sim L^2 \sim \sqrt{N}$ with finite $(\Delta - \frac{3}{2}R)$ and $Q_a - \frac{1}{2}R$ in the large N limit. In particular the duality with string theory predicts that there is a set of non-chiral primary operators, which satisfy $\Delta > \frac{3}{2}R$, whose dimension and R-charge grow without bound but such that the deviation from the BPS bound is finite in the large N limit. In the next section we will make a proposal for which operators of the gauge theory obey this peculiar scaling behavior.

Remarkably, the pp-wave geometry (2.5) that we have obtained in the scaling limit reduces to the maximally supersymmetric pp-wave solution (1.1) after performing a coordinate dependent $U(1) \times U(1)$ rotation in the $\mathbf{R}^2 \times \mathbf{R}^2$ plane as

$$\begin{aligned} z_a &= e^{ix^+} w_a \\ \bar{z}_a &= e^{-ix^+} \bar{w}_a. \end{aligned} \quad (2.9)$$

This means that the symmetry and supersymmetry of the original $AdS_5 \times T^{1,1}$ background is maximally enhanced in the Penrose limit, (2.3) and (2.4). We interpret this as saying that the corresponding subsector of the dual $\mathcal{N} = 1$ superconformal field theory has hidden $\mathcal{N} = 4$ supersymmetry.

The coordinate transformation (2.9) mapping the solution (2.5) to the maximally supersymmetric solution (1.1) allows us to write down the string Hamiltonian p^- in (2.7) in terms of the Hamiltonian $p_{S^5}^-$ of the maximally symmetric solution already computed in [6]. Using the coordinate transformation (2.9) and the relation (2.7) between the isometry and the gauge theory charges, we find that

$$\begin{aligned} \Delta - \frac{3}{2} R &= 2p^- \\ &= i \frac{\partial}{\partial x^+} \Big|_{z_a} \\ &= i \frac{\partial}{\partial x^+} \Big|_{w_a} + \sum_a \left(w_a \frac{\partial}{\partial w_a} - \bar{w}_a \frac{\partial}{\partial \bar{w}_a} \right) \\ &= 2p_{S^5}^- + J_1 + J_2, \end{aligned} \quad (2.10)$$

where J_1 and J_2 are the $U(1)$ rotation charges around an $\mathbf{R}^2 \times \mathbf{R}^2$ subspace of the maximally supersymmetric pp-wave \mathbf{R}^8 transverse space⁵, and

$$2p_{S^5}^- = \Delta_{\mathcal{N}=4} - R_{\mathcal{N}=4}. \quad (2.11)$$

We now briefly recall the string spectrum $p_{S^5}^-$ found in [6]. The spectrum consists of a set of eight bosonic harmonic oscillators a_n^i and eight fermionic harmonic oscillators S_n^α with $i, \alpha = 1, 2, \dots, 8$. The increase in light-cone energy due to one oscillator is given by

$$2\delta p_{S^5}^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2}, \quad (2.12)$$

so that in particular the zero modes a_0^i and S_0^α increase $p_{S^5}^-$ by one. To obtain the spectrum of p^- we use the twisting formula (2.10). In order to find the spectrum we need to know the charges of a_n^i and S_n^α under the $U(1) \times U(1)$ subgroup of the $SO(8)$ rotation group.

The charges of the bosonic oscillators follow from decomposing the bosonic oscillators, which transform under the $\mathbf{8}_v$ representation of the $SO(8)$ rotation group under $SU(2) \times SU(2) \times U(1) \times U(1)$, such that the \mathbf{R}^8 space on which $SO(8)$ acts splits as $\mathbf{R}^4 \times \mathbf{R}^2 \times \mathbf{R}^2$, and $U(1) \times U(1)$ rotates $\mathbf{R}^2 \times \mathbf{R}^2$. We organize the oscillators as

$$\begin{aligned} a_n^i & \quad J_1 = J_2 = 0 \quad i = 1, 2, 3, 4 \\ w_n^1 & \quad J_1 = 1, J_2 = 0 \\ \bar{w}_n^1 & \quad J_1 = -1, J_2 = 0 \\ w_n^2 & \quad J_1 = 0, J_2 = 1 \\ \bar{w}_n^2 & \quad J_1 = 0, J_2 = -1. \end{aligned} \quad (2.13)$$

Therefore, the contribution to p^- of each of the bosonic oscillators is

$$\begin{aligned} a_n^i & \quad 2\delta p^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} \\ w_n^1 & \quad 2\delta p^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} + 1 \\ \bar{w}_n^1 & \quad 2\delta p^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} - 1 \\ w_n^2 & \quad 2\delta p^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} + 1 \\ \bar{w}_n^2 & \quad 2\delta p^- = \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} - 1. \end{aligned} \quad (2.14)$$

⁵ The $U(1) \times U(1)$ rotation charges when changing from z to w coordinates remain identical.

The fermionic oscillator contribution to p^- follows by looking at the $U(1) \times U(1)$ charges carried by the $SO(8)$ spinor $\mathbf{8}_s$ under $SU(2) \times SU(2) \times U(1) \times U(1)$. The oscillators split as

$$\mathbf{8}_s \rightarrow (\mathbf{2}, \mathbf{1})_{(1/2, 1/2)} \oplus (\mathbf{2}, \mathbf{1})_{(-1/2, -1/2)} \oplus (\mathbf{1}, \mathbf{2})_{(1/2, -1/2)} \oplus (\mathbf{1}, \mathbf{2})_{(-1/2, 1/2)}, \quad (2.15)$$

where the charges in the subscript correspond to (J_1, J_2) charges. Therefore, their contribution to the light-cone Hamiltonian (2.10) is

$$\begin{aligned} S_n^{\alpha++} \quad 2\delta p^- &= \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} + 1 \\ S_n^{\alpha--} \quad 2\delta p^- &= \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} - 1 \\ S_n^{\dot{\alpha}+-} \quad 2\delta p^- &= \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} \\ S_n^{\dot{\alpha}-+} \quad 2\delta p^- &= \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2}. \end{aligned} \quad (2.16)$$

Thus we see that the spectrum of bosonic oscillators and fermionic oscillators are identical.

We note that the light-cone energy is not increased by the action of the bosonic zero mode oscillators \bar{w}_0^1 and \bar{w}_0^2 nor by the action of their supersymmetric fermionic zero mode partners $S_0^{\alpha--}$. Therefore, the system has an infinitely degenerate spectrum labeled by the number of times the vacuum state is acted on by the zero modes. In the original coordinates of the pp-wave in (2.5), the degeneracy of the spectrum can be easily understood by considering the zero-mode sector (*i.e.*, the point particle limit) of string theory. In the zero mode sector, the Hamiltonian contains, on top of four free harmonic oscillators, two decoupled Landau Hamiltonians describing an electron in a magnetic field in the planes z_1 and z_2 . The degeneracy in these coordinates corresponds to the well known Landau level degeneracy of states of the electron where the degeneracy is labeled by the angular momentum of the electron. In the appendix, we extend this to stringy excitations, quantizing the bosonic string Hamiltonian in these coordinates. We show that indeed the spectrum is twisted by $(J_1 + J_2)$ as in (2.14) with respect to the maximally supersymmetric case of [6].

In the next section we give a precise prescription of how to realize the infinite degeneracy of states in string theory, which corresponds in the dual superconformal field theory to having an infinite degeneracy of operators with a given conformal dimension, and identify the operators dual to the insertion of the string theory oscillators.

3. Gauge theory spectrum

Type IIB superstring theory in $AdS_5 \times T^{1,1}$ is dual to the $\mathcal{N} = 1$ gauge theory with gauge group $SU(N) \times SU(N)$ with two chiral multiplets A_i ($i = 1, 2$) transforming in the (N, \bar{N}) representation of the gauge group and two chiral multiplets $B_{i'}$ ($i' = 1, 2$) in the (\bar{N}, N) representation. The theory is flown to the IR fixed point. We then turn on the superpotential (1.3) involving the chiral superfields A_i and $B_{i'}$. At the fixed point, these chiral superfields have conformal weight $3/4$ and R-charge $1/2$. They transform as $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ under the $SU(2)_1 \times SU(2)_2$ global symmetry.

Now we are ready to compare states in the string theory with operators in the gauge theory. We will mainly focus only on the bosonic excitations of the theory, denoted by w_n^a and \bar{w}_n^a in (2.14). Let us begin with the zero mode sector of string theory, which is generated by \bar{w}_0^1 and \bar{w}_0^2 . Since we are dealing with the zero mode of the string, they are supergravity modes in $AdS_5 \times T^{1,1}$. The correspondence between supergravity modes and gauge theory operators has been discussed extensively in [21,22]. It is useful to rephrase it in the stringy terminology of the last section. We will find some special feature of the supergravity spectrum in the Penrose limit. Since the light-cone Hamiltonian $2p^-$ is equal to $(\Delta - \frac{3}{2}R)$, the lowest energy states of the string theory are chiral primary states. The basic ones are of the form $\text{Tr}(AB)^R$. Among them, we can identify the ground state $|0\rangle$ of the \bar{w}_0^a oscillators with the gauge theory operator,

$$|0\rangle \leftrightarrow \text{Tr} [(A_1 B_1)^R]. \quad (3.1)$$

Here and in the following, we ignore normalization factors in gauge theory operators which may depend on N and R . We have chosen A_i and $B_{i'}$ so that A_1 and A_2 carries Q_1 charge $+\frac{1}{2}$ and $-\frac{1}{2}$ and B_1 and B_2 carries Q_2 charge $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Their (J_1, J_2) charges defined by $J_a = Q_a - \frac{1}{2}R$ are therefore $(\frac{1}{4}, -\frac{1}{4})$ and $(-\frac{3}{4}, -\frac{1}{4})$ for A_1 and A_2 , and $(-\frac{1}{4}, \frac{1}{4})$ and $(-\frac{1}{4}, -\frac{3}{4})$ for B_1 and B_2 . Thus the operator $\text{Tr} [(A_1 B_1)^R]$ in (3.1) carries $J_1 = J_2 = 0$, and $\Delta - \frac{3}{2}R = \Delta_{\mathcal{N}=4} - R_{\mathcal{N}=4} = 0$. Namely it saturates the BPS bounds for both $\mathcal{N} = 1$ and $\mathcal{N} = 4$ supersymmetry algebras.

The operators \bar{w}_0^a then act on it as

$$\bar{w}_0^1 : A_1 \rightarrow A_2, \quad \bar{w}_0^2 : B_1 \rightarrow B_2. \quad (3.2)$$

In order for these to map the chiral primary state (3.1) into another chiral primary state, their action has to be symmetrized along all A_1 's and B_1 's in the trace [15]. Since the \bar{w}_0^a

oscillators are absent in the worldsheet Hamiltonian, their action does not increase $2p^- = \Delta - \frac{3}{2}R$. This is consistent with the fact that the action of the \bar{w}_0^a 's in the gauge theory gives rise to chiral primary states saturating the BPS bound of the $\mathcal{N} = 1$ supersymmetry. On the other hand, $\Delta_{\mathcal{N}=4} - R_{\mathcal{N}=4} = \Delta - \frac{3}{2}R - J_1 - J_2$ is increased by 1 every time we act with \bar{w}_0 since $J_1 + J_2 = 0$ for A_1 and B_1 while it is -1 for A_2 and B_2 . Note that $(J_1 + J_2)$ is not positive for any of the operators. Therefore these operators satisfy the BPS bounds for both the $\mathcal{N} = 1$ and the $\mathcal{N} = 4$ supersymmetry algebras. This gives an important consistency check of our conjecture about the supersymmetry enhancement.

From the way they act on A_i and $B_{i'}$, it is clear that \bar{w}_0^1, \bar{w}_0^2 and their conjugates are identified as the raising and lowering operators of the $SU(2)_1 \times SU(2)_2$ global symmetry of the gauge theory. On the string worldsheet, they act as harmonic oscillators. On the other hand, when acting on the gauge theory operators $\text{Tr}[(A_1 B_1)^R]$, they obey the constraints $(\bar{w}_0^1)^{R+1} = 0, (\bar{w}_0^2)^{R+1} = 0$. This does not contradict with the correspondence between string theory and gauge theory. Since $J_a = Q_a - \frac{1}{2}R$ ($a = 1, 2$) has to remain finite in the limit $R \rightarrow \infty$, only a finite number of \bar{w}_0 's can act on this operator and these constraints become irrelevant.

The oscillators w_0^a 's in (2.14) are more interesting. They change $(\Delta - \frac{3}{2}R)$ by 2, thus their action does not generate chiral primary states. Nevertheless the resulting states should be in the supergravity sector. Candidates for such states can be found in the list of operators given in [22], where they are called semi-conserved superfields. Although they are not chiral primaries, their conformal dimensions are protected. The ones we are interested in here take the following form,

$$\text{Tr} \left[(Ae^V \bar{A}e^{-V})^{n_1} (e^V \bar{B}e^{-V} B)^{n_2} (AB)^R \right], \quad (3.3)$$

where V is the vector multiplet for the gauge group $SU(N) \times SU(N)$. There is one important subtlety in making the identification. It was pointed out in [21] that, in order for the corresponding supergravity mode to have a rational conformal dimension, the integers n_1 and n_2 must satisfy the Diophantine equation,

$$n_1^2 + n_2^2 - 4n_1 n_2 - n_1 - n_2 = 0. \quad (3.4)$$

This is true if we are studying states with finite Δ and R . Since we are studying the scaling limit $\Delta, R \sim \sqrt{N} \rightarrow \infty$, it is worth revisiting its origin. The constraint comes from

the fact that the eigenvalue E of the Laplacian on $T^{1,1}$ for the corresponding supergravity mode takes the form

$$E = 6n_1^2 + 6n_2^2 + 8n_1n_2 + (6R + 8)(n_1 + n_2) + \frac{3}{2}R \left(\frac{3}{2}R + 4 \right). \quad (3.5)$$

One can then show that the conformal weight $\Delta = -2 + \sqrt{4 + E}$ of the mode becomes rational if (3.4) is satisfied. However, this condition is relaxed in the limit $R \rightarrow \infty$. In this limit, the meaningful quantity is $(\Delta - \frac{3}{2}R)$, and it is given by

$$\Delta - \frac{3}{2}R = 2n_1 + 2n_2 + O\left(\frac{1}{R}\right). \quad (3.6)$$

The right-hand side of this equation is clearly rational. In fact, they are even integers. Therefore the Diophantine constraint (3.4) is irrelevant in the subsector of the Hilbert space we are looking. Moreover the formula is exactly what we need to identify the action of w_0^a . Since each of these operators is supposed to increase $2p^- = \Delta - \frac{3}{2}R$ by 2 and they carry the $U(1) \times U(1)$ quantum numbers $(-1, 0)$ and $(0, -1)$, the natural identification is

$$w_0^1 : \text{insertion of } A_1 e^V \bar{A}_2 e^{-V}, \quad w_0^2 : \text{insertion of } e^V \bar{B}_2 e^{-V} B_1. \quad (3.7)$$

Of course the insertion must be symmetrized along the trace so that $(\Delta - \frac{3}{2}R)$ is minimized.

As we discussed in the last section, string theory in $AdS_5 \times T^{1,1}$ acquires enhanced $\mathcal{N} = 4$ superconformal symmetry in the Penrose limit. This means that the spectrum of the gauge theory operators in this subsector must fall into $\mathcal{N} = 4$ multiplets. Since the oscillators w_0^a and \bar{w}_0^a are part of the $\mathcal{N} = 4$ superconformal generators (P^I and J^I in the notation of [6], see also [23]), we conjecture that the chiral primary fields of the form $\text{Tr}(AB)^R$ and the semi-conserved multiplets of the form (3.3) combine to make $\mathcal{N} = 4$ multiplets in the limit.⁶ Note that this can happen only in the limit. For finite R , the semi-conserved multiplets have to obey the Diophantine constraint (3.4) in order for them to have rational conformal weights.

⁶ The decomposition of $\mathcal{N} = 4$ chiral multiplets into $\mathcal{N} = 1$ multiplets has been discussed in [24], whose results should be useful in proving this conjecture. There, the R-symmetry acting on $\mathcal{N} = 1$ multiplets is chosen to be the commutant of $SU(3)$ in the $SU(4)$ R-symmetry of the $\mathcal{N} = 4$ theory. On the other hand, the R-symmetry of the $\mathcal{N} = 1$ theory of [15] is the commutant of $SU(2) \times SU(2)$ in the $SU(4)$ R-symmetry of the enhanced $\mathcal{N} = 4$ supersymmetry.

Now let us turn to the stringy excitations. The string-bit interpretation suggests that a worldsheet operator with non-zero Fourier mode n acts on a gauge theory operator just as its $n = 0$ counter-part, but the action is summed over the trace with a position dependent phase proportional to n . This point of view was adopted in [10] to identify operators in the $\mathcal{N} = 4$ gauge theory corresponding to stringy excitations. We extend their proposal to the $\mathcal{N} = 1$ theory we are studying.

Consider the oscillators \bar{w}_n^a . When $n = 0$, it is defined as the replacement $A_1 \rightarrow A_2$ for $a = 1$ and $B_1 \rightarrow B_2$ for $a = 2$, averaged over the trace. It is then natural to identify

$$\begin{aligned}\bar{w}_n^1|0\rangle &\leftrightarrow \sum_{k=0}^{R-1} \text{Tr} [(A_1 B_1)^k A_2 B_1 (A_1 B_1)^{R-1-k}] e^{\frac{2\pi i n k}{R}} \\ \bar{w}_n^2|0\rangle &\leftrightarrow \sum_{k=0}^{R-1} \text{Tr} [(A_1 B_1)^k A_1 B_2 (A_1 B_1)^{R-1-k}] e^{\frac{2\pi i n k}{R}}.\end{aligned}\tag{3.8}$$

Of course, the operators on the right-hand side vanish due to the cyclicity of the trace. This corresponds to the string theory fact that the left-hand side does not satisfy the level matching momentum constraint, as explained in [10]. The idea is to use an analogous definition for $\bar{w}_{n_1}^1 \cdots \bar{w}_{n_s}^1 \bar{w}_{m_1}^2 \cdots \bar{w}_{m_t}^2|0\rangle$ such that $\sum n_i + \sum m_{i'} = 0$. For example,

$$\bar{w}_{-n}^1 \bar{w}_n^2|0\rangle \leftrightarrow \sum_{k=0}^{R-1} \text{Tr} [A_2 (B_1 A_1)^k B_2 (A_1 B_1)^{R-1-k}] e^{\frac{2\pi i n k}{R}}.\tag{3.9}$$

Operators of this type are not chiral primaries. As pointed out in [15], these operators vanish if we use the constraint due to the superpotential (1.3),

$$A_1 B_{i'} A_2 = A_2 B_{i'} A_1, \quad B_1 A_i B_2 = B_2 A_i B_1.\tag{3.10}$$

Since they are defined against the potential wall, they gain additional conformal weights beyond their naive values, and therefore $\Delta - \frac{3}{2}R > 0$. The string theory computation in section 3 predicts that the action of \bar{w}_n changes $(\Delta - \frac{3}{2}R)$ by

$$\begin{aligned}\delta \left(\Delta - \frac{3}{2}R \right)_n &= \sqrt{1 + \left(\frac{n}{\alpha' p^+} \right)^2} - 1 \\ &= \sqrt{1 + 3\pi g_s n^2 \frac{N}{R^2}} - 1.\end{aligned}\tag{3.11}$$

We are taking the large N limit so that N/R^2 remains finite. When $g_s > 0$, the right-hand side is indeed strictly positive. One of the interesting features of this formula is

that $\delta(\Delta - \frac{3}{2}R)_n$ vanishes at $g_s = 0$ even for $n \neq 0$, giving rise to further degeneracy of the spectrum. In the large N limit, the only parameter of the string theory is g_s while that of the gauge theory is λ , the coefficient of the superpotential. Since the superpotential must increase the conformal weights of these operators, one explanation of what happens at $g_s = 0$ is that λ vanishes in the gauge theory side. To our knowledge, a map between these parameters has not been worked out, and this observation may give us some hint about the correspondence between the gauge theory moduli and the string theory moduli. At $g_s = 0$, the string theory Hilbert space becomes the Fock space of first quantized strings, whose spectrum is integral. It would be interesting to see whether such a structure emerges in this subsector of the gauge theory at $\lambda = 0$.

The other set of operators w_n^a are similarly interpreted in the gauge theory side. These operators insert $Ae^V \bar{A}e^{-V}$ and $e^V \bar{B}e^{-V} B$ in the trace and sum over insertion points along the trace with position dependent phases. The string theory computation predicts that the amount of change of $(\Delta - \frac{3}{2}R)$ is given by

$$\begin{aligned} \delta \left(\Delta - \frac{3}{2}R \right)_n &= \sqrt{1 + \left(\frac{n}{\alpha' p^+} \right)^2} + 1 \\ &= \sqrt{1 + 3\pi g_s n^2 \frac{N}{R^2}} + 1. \end{aligned} \tag{3.12}$$

One can also consider operators corresponding to stringy excitations in the r^i directions. In [10], these are interpreted as taking derivative of operators with respect to spatial coordinates in the gauge theory. In our case, one may be puzzled by the fact that there seem to be two types of derivatives, those acting on A 's and those acting on B 's. Clearly only particular combinations of them correspond to stringy excitations of the r^i directions. In general, there are many gauge invariant observables one can write down, and only some of them correspond to string states. We expect that the others become infinitely heavy, *i.e.* $(\Delta - \frac{3}{2}R)$ becomes infinitely large, in the large N limit.

4. Other examples

The Penrose limit focuses on the geometry near a null geodesic. When we have a gauge theory whose supersymmetry is reduced by placing branes on a curved space, the Penrose limit may flatten out the space near the branes and restore supersymmetry. Thus

we expect that enhancement of symmetry takes place in a large class of theories which have gravity duals.

Let us consider the supergravity solution found in [16] which is dual to pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (with Kaluza-Klein tower of fields). We will show that it has a Penrose limit which is identical to that of a collection of five-branes in flat space. Thus, in this case, symmetry is again enhanced in the corresponding subsector of $\mathcal{N} = 1$ super Yang-Mills. To exhibit this we write the metric of NS-NS five-branes in flat space

$$ds^2 = ds^2(R^{1,5}) + L^2 \left(d\rho^2 + \frac{1}{4}(d\psi + \cos\theta d\phi)^2 + \frac{1}{4}d\theta^2 + \frac{1}{4}\sin^2\theta d\phi^2 \right), \quad (4.1)$$

where $L^2 = \alpha' N$. We introduce the following null coordinates

$$\begin{aligned} x^+ &= \frac{1}{2} \left(\frac{t}{L} + \frac{1}{2}(\psi + \phi) \right) \\ x^- &= \frac{L^2}{2} \left(\frac{t}{L} - \frac{1}{2}(\psi + \phi) \right) \end{aligned} \quad (4.2)$$

and consider the limit around $\rho = \theta = 0$. We take the limit $L \rightarrow \infty$ while rescaling the coordinates

$$\rho = \frac{r}{L} \quad \theta = \frac{2y}{L}. \quad (4.3)$$

The metric that one obtains in this limit is

$$ds^2 = -4dx^+dx^- - 2y^2d\phi dx^+ + ds^2(\mathbf{R}^8), \quad (4.4)$$

which by using the coordinate transformation in (2.9) reduces to the pp-wave metric

$$ds^2 = -4dx^+dx^- - w\bar{w}dx^+dx^+ + dw d\bar{w} + ds^2(\mathbf{R}^6), \quad (4.5)$$

where w is a complex coordinate on an \mathbf{R}^2 plane in \mathbf{R}^8 . Moreover, in the limit (4.4) the dilaton becomes constant and the NS-NS three-form flux $H_{+12} = \text{const}$ becomes null. The worldsheet theory describing the coordinates (x^+, x^-, w, \bar{w}) can be identified with the WZW model based on the non-semi-simple group which is the central extension of the two-dimensional Poincare group, found by Nappi and Witten [17].

One can also show that the Penrose limit of the Maldacena-Nuñez solution [16] in the region near $\rho = 0$ gives rise to a generalization of the Nappi-Witten geometry with

16 supercharges and again the supersymmetry is enhanced.⁷ It would be interesting to explore the consequences of this symmetry enhancement for the gauge theory.

Another interesting example is to consider the Penrose limit of the dual pair obtained by placing N D3-branes at a C^3/Z_3 orbifold singularity. The gauge theory living on the D3-branes is an $\mathcal{N} = 1$ four-dimensional quiver gauge theory [25] and the gravity dual is $AdS_5 \times S^5/Z_3$ [26], where the S^5 is described by the complex coordinates z_i ($i = 1, 2, 3$) constrained by

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \quad (4.6)$$

and the Z_3 generator g acts by

$$g \cdot z_i = \alpha z_i \quad i = 1, 2, 3 \quad \alpha^3 = 1, \quad (4.7)$$

so that Z_3 acts freely on the sphere. The $U(1)_R$ symmetry of the gauge theory can be identified with shifts along the Hopf fiber coordinate ψ when S^5/Z_3 is described as a $U(1)$ bundle over CP^2

$$ds^2 = (d\psi + \sin^2 \mu \sigma_3)^2 + d\mu^2 + \sin^2 \mu (\sigma_1^2 + \sigma_2^2 + \cos^2 \mu \sigma_3^2), \quad (4.8)$$

where σ_i are a set of left-invariant $SU(2)$ one forms satisfying $d\sigma_i = \epsilon_{ijk} \sigma_j \wedge \sigma_k$. The Z_3 orbifold group acts by restricting the range of the ψ coordinate to $1/3$ of the usual value for S^5 while leaving all other coordinates intact. We introduce null coordinates

$$\begin{aligned} x^+ &= \frac{1}{2} (t + \psi) \\ x^- &= \frac{L^2}{2} (t - \psi) \end{aligned} \quad (4.9)$$

taking the $L \rightarrow \infty$ near $\mu = \rho = 0$ with⁸

$$\rho = \frac{r}{L} \quad \mu = \frac{y}{L}. \quad (4.10)$$

⁷ In the earlier version of this paper, we claimed that the Penrose limit of the Maldacena-Nuñez solution is identical to the Nappi-Witten geometry (4.5). We thank Juan Maldacena and Horatiu Nastase for pointing out an error in our argument. Our observation about the enhancement of supersymmetry in this case still remains true.

⁸ The ρ coordinate comes the AdS_5 part of the metric.

In the limit one gets

$$ds^2 = -4dx^+ dx^- + \sum_{i=1}^4 (dr^i dr^i - r^i r^i dx^+ dx^+ + dy^i dy^i) + 2y^2 dx^+ \sigma_3, \quad (4.11)$$

where y_i are Cartesian coordinates in \mathbf{R}^4 . By introducing a pair of complex coordinates z_a for \mathbf{R}^4 one can show that (4.11) can be rewritten as

$$ds^2 = -4dx^+ dx^- + \sum_{i=1}^4 (dr^i dr^i - r^i r^i dx^+ dx^+) + \sum_a (dz_a d\bar{z}_a + i(z_a d\bar{z}_a - \bar{z}_a dz_a) dx^+), \quad (4.12)$$

which we have already shown is the same as the maximally supersymmetric pp-wave metric (1.1). So we have another example in which a subsector of a four dimensional $\mathcal{N} = 1$ gauge theory is enhanced to $\mathcal{N} = 4$. It would be interesting to match the string oscillators in this background with operators in the quiver gauge theory.

We should note, however, that the Penrose limit does not always enhance supersymmetry. For example, we can consider the dual pair generated by placing a collection of D3-branes at a C^2/Z_2 orbifold singularity. The gauge theory is a four dimensional $\mathcal{N} = 2$ gauge theory [25] and the gravity dual is $AdS_5 \times S^5/Z_2$ [26], where the S^5 is described by (4.6) and the Z_2 generator g acts by

$$g \cdot z_1 = -z_1 \quad g \cdot z_2 = -z_2. \quad (4.13)$$

Thus the Z_2 action has as fixed locus an S^1 described by $|z_3|^2 = 1$ at $z_1 = z_2 = 0$. The coordinate parametrizing the fixed S^1 can be identified with the ψ coordinate considered in the Penrose limit of $AdS_5 \times S^5$ [10]. Therefore, the corresponding pp-wave limit is given by the Z_2 orbifold of the maximally supersymmetric pp-wave (1.1), where Z_2 acts on an \mathbf{R}^4 subspace of the transverse \mathbf{R}^8 space. The main difference between this and the previous orbifold is that ψ is not acted by Z_2 while the other angles on the S^5 have Z_2 identifications. This is related to the fact that the S^5 has a fixed circle in this case.

To find the amount of supersymmetry left unbroken by the orbifold, one must find which components of the Killing spinors of the pp-wave geometry (1.1) are left invariant under the Z_2 action. The Killing spinors of (1.1) were found [7] and take the form

$$\epsilon(x, y, x^+) = f(x, y, x^+) \epsilon_0, \quad (4.14)$$

where x and y are the transverse $\mathbf{R}^4 \times \mathbf{R}^4$ coordinates, f is a function which can be found in [7] and ϵ_0 is a constant $SO(8)$ spinor. The amount of unbroken supersymmetry is the number of Killing spinors (4.14) left invariant under the Z_2 action. Therefore, the unbroken supersymmetries satisfy

$$\begin{aligned}\epsilon(x, y, x^+) &= g \cdot \epsilon(x, y, x^+) \\ &= \gamma^{5678} \epsilon(x, -y, x^+) \\ &= f(x, y, x^+) \gamma^{5678} \epsilon_0,\end{aligned}\tag{4.15}$$

namely,

$$\gamma^{5678} \epsilon_0 = \epsilon_0.\tag{4.16}$$

Therefore the orbifold preserves 1/2 of the supersymmetry which is generated by Killing spinors satisfying this condition.

5. Discussion

In this paper we have given an explicit example of an $\mathcal{N} = 1$ superconformal field theory which, in the large N limit, has a subsector of the Hilbert space with enhanced $\mathcal{N} = 4$ superconformal symmetry. We have arrived at this perhaps unexpected conclusion by taking the corresponding limit in the string theory side and by showing that it becomes identical to the theory with higher supersymmetry. The subsector of the gauge theory that should exhibit this symmetry enhancement is dictated by the Penrose limit which restricts the space of states of the gauge theory to those whose conformal dimension and R-charge diverge in the large N limit but which nevertheless have finite $(\Delta - \frac{3}{2}R)$.

The light-cone Hamiltonian for the background that one obtains in the limit can be found from the Hamiltonian of [6] for the maximally supersymmetric case by twisting it with $U(1)$ charges. In this way we get a prediction for the spectrum of $(\Delta - \frac{3}{2}R)$ of the $\mathcal{N} = 1$ superconformal field theory. We proposed how stringy excitations are related to gauge theory operators and made predictions about the gauge theory spectrum. Perhaps the most striking one is that the various $\mathcal{N} = 1$ multiplets should turn into multiplets of $\mathcal{N} = 4$ supersymmetry. In particular, the chiral multiplets and the semi-conserved multiplets of $\mathcal{N} = 1$ supersymmetry should combine into $\mathcal{N} = 4$ chiral multiplets. It would be interesting to explore this prediction further. The duality also predicts values of $(\Delta - \frac{3}{2}R)$ for certain operators in this strongly coupled gauge theory.

We have shown that the enhancement of symmetry in the Penrose limit is a fairly generic phenomenon in theories which have gravity duals. One can intuitively understand this as due to the fact that the limit flattens out parts of spacetime by focusing on a region near a null geodesic. For example, we found that the Penrose limit of a collection of flat NS-NS five-branes is the Nappi-Witten geometry. The limit of the Maldacena-Nuñez geometry is its variation and also has 16 supercharges. This is an interesting case since we can quantize the worldsheet theory without taking the light-cone gauge, and we can compute correlation functions and other observables using the standard techniques of conformal field theory [27]. On the other hand, we also found cases in which the Penrose limit does not lead to symmetry enhancement. It would be interesting to explore further the consequences of this enhancement of symmetries for QCD-like theories and see which lessons this might teach us for the familiar questions about strongly coupled dynamics of these gauge theories.

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Appendix A. String spectrum in the original coordinates

It may be instructive to show how we can quantize string theory using the metric (2.5) before we make the coordinate transformation to its manifestly symmetric form (1.1). We can read off the bosonic part of the light-cone action from the metric (2.5) as

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2} \sum_{i=1}^4 (\dot{r}_i^2 - r_i'^2) + \sum_{a=1,2} \left(\frac{1}{2} (\dot{x}_a^2 + \dot{y}_a^2 - x_a'^2 - y_a'^2) - x_a \dot{y}_a + y_a \dot{x}_a \right) \right], \quad (\text{A.1})$$

where $\dot{} = \partial_\tau$ and $' = \partial_\sigma$.

The spectrum of the r_i part of the light-cone Hamiltonian is

$$H_{r\text{-part}} = \sum_{n=-\infty}^{\infty} N_n^{(r)} \sqrt{1 + \left(\frac{n}{\alpha' p^+} \right)^2}, \quad (\text{A.2})$$

as in the case of the $\mathcal{N} = 4$ theory in [10], where n is the label of the Fourier mode around the σ direction and $N_n^{(r)}$ is the number of excitations of that mode.

Let us examine the (x_a, y_a) part of the Hamiltonian. In the following, we will ignore the index a with the assumption that we are referring to either $a = 1$ or 2 . Accordingly the manifest global symmetry is $U(1) \subset SU(2)$. In the Fourier expansion,

$$x = \frac{1}{\sqrt{p^+}} \sum_n x_n e^{i \frac{n}{\alpha' p^+} \sigma}, \quad y = \frac{1}{\sqrt{p^+}} \sum_n y_n e^{i \frac{n}{\alpha' p^+} \sigma}, \quad (\text{A.3})$$

the Hamiltonian becomes

$$H_{xy\text{-part}} = \sum_{n=0}^{\infty} H_n, \quad (\text{A.4})$$

where

$$H_{n \neq 0} = (p_{x_n} - y_{-n}) (p_{x_{-n}} - y_n) + (p_{y_n} + x_{-n}) (p_{y_{-n}} + x_n) + \left(\frac{n}{\alpha' p^+} \right)^2 (x_n x_{-n} + y_n y_{-n}), \quad (\text{A.5})$$

and

$$H_0 = \frac{1}{2} (p_{x_0} - y_0)^2 + \frac{1}{2} (p_{y_0} + x_0)^2, \quad (\text{A.6})$$

where

$$p_{x_n} = -i \frac{\partial}{\partial x_n}, \quad p_{y_n} = -i \frac{\partial}{\partial y_n}. \quad (\text{A.7})$$

To compare with the gauge theory spectrum, it is useful to use the complex coordinates, $z_n = x_n + iy_n$ and $\bar{z}_n = x_n - iy_n$, so that the $U(1)$ part of the global $SU(2)$ symmetry is manifest. The Hamiltonians for the Fourier modes then become

$$H_{n \neq 0} = \left(p_{\bar{z}_{-n}} + \frac{i}{2} z_n \right) \left(p_{z_n} - \frac{i}{2} \bar{z}_{-n} \right) + \left(p_{\bar{z}_n} - \frac{i}{2} z_{-n} \right) \left(p_{z_{-n}} + \frac{i}{2} \bar{z}_n \right) + \frac{1}{2} \left(\frac{n}{\alpha' p^+} \right)^2 (z_n \bar{z}_{-n} + z_{-n} \bar{z}_n), \quad (\text{A.8})$$

and

$$H_0 = \left(p_{\bar{z}_0} + \frac{i}{2} z_0 \right) \left(p_{z_0} - \frac{i}{2} \bar{z}_0 \right). \quad (\text{A.9})$$

Let us diagonalize them.

(a) Zero mode

The Hamiltonian H_0 for the zero mode is nothing but the one for a charged particle in two dimensions in a constant magnetic field. It has the Landau spectrum with infinite degeneracy at each level. To compare with the gauge theory spectrum, it is useful to introduce the following set of oscillators,

$$\begin{aligned} a_0 &= \frac{1}{2} \bar{z}_0 + \frac{\partial}{\partial z_0}, & a_0^\dagger &= \frac{1}{2} z_0 - \frac{\partial}{\partial \bar{z}_0}, \\ b_0 &= \frac{1}{2} z_0 + \frac{\partial}{\partial \bar{z}_0}, & b_0^\dagger &= \frac{1}{2} \bar{z}_0 - \frac{\partial}{\partial z_0}. \end{aligned} \quad (\text{A.10})$$

They obey the commutation relations

$$[a_0, a_0^\dagger] = 1, \quad [b_0, b_0^\dagger] = 1, \quad (\text{others}) = 0. \quad (\text{A.11})$$

Under the $U(1)$ subgroup of the $SU(2)$ global symmetry, the operators a_0, b_0^\dagger carry charge -1 and a_0^\dagger, b_0 carry charge $+1$

In terms of the oscillators, the Hamiltonian (A.9) is expressed as

$$H_0 = 2a_0^\dagger a_0 + 1. \quad (\text{A.12})$$

In particular, it commutes with the oscillators b_0, b_0^\dagger . The lowest energy states are annihilated by a_0 and thus have the form $(b_0^\dagger)^k \exp(-\frac{1}{2} z_0 \bar{z}_0)$ where k is any non-negative integer. The complete energy eigenstates are given by

$$\psi_{m,k}^{(0)} = (a_0^\dagger)^m (b_0^\dagger)^k \exp\left(-\frac{1}{2} z_0 \bar{z}_0\right), \quad (\text{A.13})$$

with $m, k \geq 0$. The energy of the state is $2m$ and the $U(1)$ global charge is $(k - m)$.

(b) Non-zero modes

As in the case of the zero mode, we introduce the following set of oscillators,

$$\begin{aligned} a_n &= \frac{1}{2}\bar{z}_n + \frac{\partial}{\partial z_{-n}}, & a_n^\dagger &= \frac{1}{2}z_{-n} - \frac{\partial}{\partial \bar{z}_n}, \\ b_n &= \frac{1}{2}z_n + \frac{\partial}{\partial \bar{z}_{-n}}, & b_n^\dagger &= \frac{1}{2}\bar{z}_{-n} - \frac{\partial}{\partial z_n}, \end{aligned} \quad (\text{A.14})$$

obeying the commutation relations,

$$[a_n, a_n^\dagger] = 1, \quad [b_n, b_n^\dagger] = 1, \quad (\text{others}) = 0. \quad (\text{A.15})$$

In the limit $n^2/\alpha'p^+ \rightarrow 0$, the Hamiltonian H_n takes the same form as in the case of the zero mode,

$$H_n = 2a_n^\dagger a_n + 2a_{-n}^\dagger a_{-n} + 2. \quad (\text{A.16})$$

In particular, each energy level has infinite degeneracy generated by the oscillators $b_{\pm n}^\dagger$.

For finite $n^2/\alpha'p^+$, the Hamiltonian H_n contains terms mixing the two oscillators,

$$\begin{aligned} H_n &= 2a_n^\dagger a_n + \frac{1}{2}\omega^2(a_n^\dagger + b_{-n})(a_n + b_{-n}^\dagger) \\ &\quad + (n \rightarrow -n) + 2, \end{aligned} \quad (\text{A.17})$$

where we introduced $\omega = n/\alpha'p^+$ to simplify the following equations. The Hamiltonian can be diagonalized by introducing a new set of oscillators α_n and β_n defined by

$$\begin{aligned} a_n &= \cosh \theta \alpha_n + \sinh \theta \beta_{-n}^\dagger, & a_n^\dagger &= \cosh \theta \alpha_n^\dagger + \sinh \theta \beta_{-n}, \\ b_n &= \cosh \theta \beta_n + \sinh \theta \alpha_{-n}^\dagger, & b_n^\dagger &= \cosh \theta \beta_n^\dagger + \sinh \theta \alpha_{-n}. \end{aligned} \quad (\text{A.18})$$

They obey the commutation relations

$$[\alpha_n, \alpha_n^\dagger] = 1, \quad [\beta_n, \beta_n^\dagger] = 1, \quad (\text{others}) = 0. \quad (\text{A.19})$$

The vacuum state for α, β is related to the one for a, b by the Bogolubov transformation. Substituting these into (A.17) and requiring that the cross terms of α and β to vanish, we find

$$e^{-4\theta} = 1 + \omega^2. \quad (\text{A.20})$$

In terms of the new set of oscillators, the Hamiltonian H_n is then expressed as

$$H_n = \left(\sqrt{1 + \omega^2} + 1 \right) \alpha_n^\dagger \alpha_n + \left(\sqrt{1 + \omega^2} - 1 \right) \beta_n^\dagger \beta_n + (n \rightarrow -n) + 2\sqrt{1 + \omega^2}. \quad (\text{A.21})$$

Here we have chosen a solution so that, in the limit $\omega \rightarrow 0$, the oscillators become $\alpha_n \rightarrow a_n$ and $\beta_n \rightarrow b_n$. Energy eigenstates of H_n are then given by $(\alpha_n^\dagger)^m (\alpha_{-n}^\dagger)^{m'} (\beta_n^\dagger)^k (\beta_{-n}^\dagger)^{k'} |0\rangle$, where $|0\rangle$ is the vacuum state for the α, β oscillators. The state carries the energy,

$$E_{m,m',k,k'} = (m + m') \left(\sqrt{1 + \omega^2} + 1 \right) + (k + k') \left(\sqrt{1 + \omega^2} - 1 \right),$$

and the global U(1) charge $(k + k' - m - m')$.

Combining this with the result for the zero mode, the complete spectrum for the (x, y) part of the light-cone Hamiltonian is given by

$$H_{xy-part} = \sum_{n=-\infty}^{\infty} \left[N_n^{(\alpha)} \left(\sqrt{1 + \left(\frac{n}{\alpha' p^+} \right)^2} + 1 \right) + N_n^{(\beta)} \left(\sqrt{1 + \left(\frac{n}{\alpha' p^+} \right)^2} - 1 \right) \right]. \quad (\text{A.22})$$

This result agrees with the one we obtained in section 2, with the identification:

$$\alpha_n = \bar{w}_n, \quad \beta_n = w_n \quad (\text{A.23})$$

in (2.14).

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